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PHYSICAL STUDY OF STEADY ELECTROMAGNETOFLUID-DYNAMIC VISCOUS FLOWS

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Abstract—This paper presents a geometric study and solutions of the electromagnetofluid-dynamic (EMFD) flows. On the basis of the reduced fundamental equations the fascination of charge, solutions and circulation-preserving EMFD flows have been discussed.

1. INTRODUCTION

The symbolic relationship that exists between mathematics and its applications has been a long and fruitful one, especially in the fields of physics and engineering. The most successful mathematical models of phenomena (whether they be in physics, biology, the social sciences, or other areas) appear to be those which are capable of identifying and sustaining complementary levels of description. The general problem of electromagnetofluid-dynamics (EMFD) is quite complex. Kingston and Power [1] gave an elegant analysis of the dynamics of two-dimensional aligned flows. This analysis establishes that if the charge density is not identically zero in a flow region, then the magnetic field must be irrotational everywhere. Babu and Singh [2] have discussed hodograph transformations of rotational EMFD flows and obtained some solutions. However, Singh and Singh [3, 4], Singh [5], Singh and Tripathi [6] have discussed the geometry of MFD flows and circulation-preserving motion.

The aim of this paper is to study steady EMFD flows when the charge density is not identically zero. This work is carried out by employing a well-established fluid-dynamic technique of establishing integrability conditions for scalar fields. The fascination of charge distribution and circulation-preserving motion have been considered.

The plan of this work is as follows: in Section 2 we recepitulate the basic equations governing the steady motion of two-dimensional EMFD flows when the charge density is not identically zero. Section 3 deals the circulation. Sections 4 and 5 contains the formulation of the flow equations in curvilinear coordinates and the realization of curvilinear coordinates. Section 6 deals the radial flows and lastly, in Section 7, a theorem on the basis of the result of Section 3 has been established and the geometric studies have been undertaken.

2. FLOW EQUATIONS

The fundamental EMFD equations, for the steady flow of a viscous incompressible fluid are [7]:

$$\operatorname{div} \bar{v} = 0, \tag{1}$$

$$\rho(\bar{v}\cdot\operatorname{grad}) v + \operatorname{grad} p = \mu J \times \bar{H} + \eta \nabla^2 \bar{v} + q\bar{E}; \tag{2}$$

$$J = I + q\bar{v} = \operatorname{curl} \bar{H}; \tag{3}$$

$$I = \sigma(E + \mu\bar{v} \times \bar{H}); \tag{4}$$

$$\operatorname{curl} \bar{E} = 0; \tag{5}$$

$$\operatorname{div} \bar{E} = q/\epsilon \tag{6}$$

and

$$\operatorname{div} \bar{H} = 0, \tag{7}$$

where $\bar{v}$, $\bar{H}$, $\bar{E}$, $\bar{J}$, q and p are the velocity field, the magnetic field, the electric field, the current density field, the charge density function and the pressure function. Here ρ, σ, η, μ and ϵ are respectively, the constant density, the constant electric conductivity, the constant coefficient of viscosity, the constant magnetic permeability and the constant permitivity of the fluid.

Following Babu and Singh [2], if the steady plane EMFD flow has non-zero charge density and the magnetic field is in the flow plane, then the vector fields $\bar{v}$ and $\bar{H}$ satisfy

$$\operatorname{curl} \bar{H} = \mu\sigma\bar{v} \times \bar{H} \tag{8}$$

and the flow is governed by the system of equations

$$\operatorname{div} \bar{v} = 0, \tag{9}$$

$$\rho(\bar{v}\cdot\operatorname{grad})\bar{v} + \operatorname{grad} p = \mu(\operatorname{curl} \bar{H}) \times \bar{H} + \eta\nabla^2\bar{v} - \frac{q^2}{\sigma}\bar{v} \tag{10}$$

$$\operatorname{div}(q\bar{v}) = -\frac{\sigma}{\epsilon}q, \tag{11}$$

$$\operatorname{curl}(q\bar{v}) = \bar{0}, \tag{12}$$

$$\operatorname{curl} \bar{H} = \mu\sigma\bar{v} \times \bar{H}, \tag{13}$$

$$\operatorname{div} \bar{H} = 0, \tag{14}$$

$$\bar{E} = -\frac{q}{\sigma}\bar{v} \tag{15}$$

and

$$I = \operatorname{curl} \bar{H} - q\bar{v}. \tag{16}$$

Equations (11) and (12), respectively, can be written, by using equation (9), as

$$\operatorname{grad}(\ln q)\cdot\bar{v} + \frac{\sigma}{\epsilon} = 0, \tag{17}$$

$$\operatorname{grad}(\ln q) \times \bar{v} + \operatorname{curl} \bar{v} = 0. \tag{18}$$

Taking the vector product of equation (18) with $\bar{v}$ and using equation (17), we obtain

$$\operatorname{grad}(\ln q) = -\left[\frac{\frac{\sigma}{\epsilon}\bar{v} + \bar{v} \times \operatorname{curl} \bar{v}}{\bar{v}\cdot\bar{v}}\right]. \tag{19}$$

Taking the curl of equation (19), it follows that

$$\operatorname{curl}\left[\frac{(\operatorname{curl} \bar{v}) \times \bar{v} - \frac{\sigma}{\epsilon}\bar{v}}{\bar{v}\cdot\bar{v}}\right] = 0, \tag{20}$$

which is the integrability condition for the function $\ln q$. The integrability condition (20) is satisfied by every steady plane incompressible EMFD flow. However, if the flows are such that $\bar{H}$ and $\bar{v}$ are every where parallel (aligned flows), we obtain one more integrability condition.

From the definition of aligned flows, we have

$$\bar{H} = F(x, y)\bar{v}, \tag{21}$$

where $F(x, y)$ is some arbitrary scalar function satisfying, by use of equations (9) and (14), the condition

$$\bar{v}\cdot\operatorname{grad} F = 0. \tag{22}$$

Employing equations (21) in (13), we have

$$\operatorname{grad}(\ln F) \times \bar{v} + \operatorname{curl} \bar{v} = 0. \tag{23}$$

Taking the vector product of this equation with $\bar{v}$ and using equation (22), we obtain

$$\operatorname{grad}(\ln F) = \frac{(\operatorname{curl} \bar{v}) \times \bar{v}}{\bar{v} \cdot \bar{v}} \tag{24}$$

and therefore, get the integrability condition for $\ln F$ given by

$$\operatorname{curl}\left[\frac{(\operatorname{curl} \bar{v}) \times \bar{v}}{\bar{v} \cdot \bar{v}}\right] = \bar{0}. \tag{25}$$

It follows from equations (20) and (25) that for plane aligned flows, the vector field $\bar{v}$ satisfies

$$\operatorname{curl}\left[\frac{\bar{v}}{\bar{v} \cdot \bar{v}}\right] = 0, \tag{26}$$

and equation (25). Kingston and Power [1] obtained equation (26), in their study of inviscid aligned flows, by a complex variable technique. However, their derivation of equation (26) requires $F(x, y)$ to be a constant throughout the flow.

3. CIRCULATION

Let C be any arbitrary closed curve which moves with the fluid and bounds an area A. By definition of circulation Γ around C, we have

$$\Gamma = \int_c \bar{v} \, \mathrm{d}l$$

and, therefore,

$$\frac{\mathrm{D}}{\mathrm{D}t}\Gamma = \int_c \bar{a} \cdot \mathrm{d}l = \int_A \int \operatorname{curl} \bar{a} \cdot \mathrm{d}s = \int_A \int \operatorname{curl}(\xi \times \bar{v}) \cdot \mathrm{d}s, \tag{27}$$

where $\bar{a} = (\bar{v} \cdot \nabla)\bar{v}$ is the acceleration vector field.

From this equation we see that the circulation Γ, for any closed curve C moving with the fluid, is conserved if and only if $\operatorname{curl}(\bar{\xi} \times \bar{v})$ vanishes identically. However, from the integrability condition (25) we find, with the aid of equation (26), that

$$\operatorname{curl}(\bar{\xi} \times \bar{v}) = 0, \tag{28}$$

if and only if $(\operatorname{grad}|\bar{v}| \cdot \bar{v})\bar{\xi} = 0$. Thus we are led to the result that the circulation is conserved when either the flow is irrotational or the velocity magnitude is constant on any individual stream-line, and conversely.

Equation (28) is itself a necessary and sufficient condition for a steady motion to be circulation-preserving [8]. The problem of circulation-preserving MFD flows with lamellar vorticity has been completely solved by Singh and Singh [4].

4. FORMULATION OF THE FLOW EQUATIONS IN CURVILINEAR COORDINATES

We now transform the flow equations to a system of curvilinear coordinates (ϕ, ψ) by virtue of the transformation (Fig. 1)

$$x = f(\phi, \psi), \quad y = g(\phi, \psi). \tag{29}$$

We assume quite generally that this curvilinear system is orthogonal. The metric differential form can then be written as

$$\mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}s^2 = E(\phi, \psi)\,\mathrm{d}\phi^2 + G(\phi, \psi)\,\mathrm{d}\psi^2, \tag{30}$$

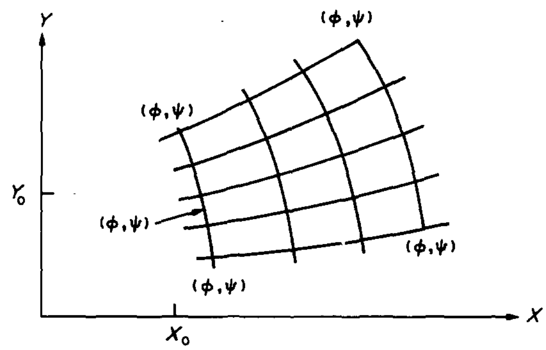

Fig. 1. Curvilinear system of coordinates.

where the tensor components E and G are defined by

$$E = f_\phi^2 + g_\phi^2, \quad G = f_\psi^2 + g_\psi^2. \tag{31}$$

The orthogonality of the curvilinear system is expressed by

$$F = f_\phi f_\psi + g_\phi g_\psi = 0. \tag{32}$$

Equations (31) and (32) yield some relations which will be used in the following:

$$f_\phi = J\psi_g, \quad g_\phi = -J\psi_f, \quad \psi_f = -J\phi_g, \quad g_\psi = J\phi_f, \tag{33}$$

provided that $0 < |J| < \infty$, where J denotes the Jacobian

$$J = f_\phi g_\psi - f_\psi g_\phi. \tag{34}$$

From equations (31) and (34), we have

$$J = \pm W, \tag{35}$$

where

$$W = f_\phi g_\psi - g_\phi f_\psi = \sqrt{EG - F^2} = \frac{1}{k}, \quad k \neq 0,$$

$$f_{\phi\phi} f_\psi + g_{\phi\phi} g_\psi = F_\phi - \tfrac{1}{2}E = -\tfrac{1}{2}E_\psi,$$
$$f_{\psi\psi} f_\phi + g_{\psi\psi} g_\phi = F_\psi - \tfrac{1}{2}G_\phi = -\tfrac{1}{2}G_\phi. \tag{36}$$

Let $a(\phi, \psi)$ and $b(\phi, \psi)$ denote the components of the particle velocity in the direction of ϕ and ψ, respectively, i.e. the projection of $\bar{v}$ into the tangent unit vectors of the lines ψ = constant and ϕ = constant at the point (ϕ, ψ). It should be noted that no distinction between the contravariant and covariant components needs to be made since the curvilinear system is orthogonal. We now assume quite generally that the ϕ-lines, i.e. the lines ψ = constant, constitute the streamlines. With this assumption, the vector component b can be taken identically zero. The magnitude of the particle velocity is then given by

$$\bar{v} = \sqrt{v_1^2 + v_2^2} = a. \tag{37}$$

Equations (1) and (7), respectively imply the existence of a stream function ψ and the magnetic function ϕ, such that

$$-\psi_x = v_2, \quad \psi_y = v_1 \tag{38}$$

and

$$\phi_x = H_2, \quad -\phi_y = H_1. \tag{39}$$

From equations (31), (33), (37) and (38), we obtain

$$v = \frac{\sqrt{E}}{J}, \quad H = \frac{\sqrt{G}}{J}. \tag{40}$$

Substituting equations (31)–(35), (38)–(40) in flow equations (1)–(7), we obtain

$$h_\psi + \rho\xi = \frac{\eta}{J} G\rho\phi, \tag{41}$$

$$h_\phi + \mu\Omega = \frac{\eta}{J} E\rho - \frac{E}{J}\frac{g^2}{\sigma}, \tag{42}$$

$$(\ln q)_\phi = -\frac{\sigma}{\epsilon} J, \tag{43}$$

$$(\ln q)_\psi = \xi \frac{J^2}{E}, \tag{44}$$

$$\Omega = \frac{\mu\sigma}{J} = \frac{1}{J}\left(\frac{G}{J}\right)_\phi, \tag{45}$$

$$\bar{\xi} = -\frac{1}{J}\left(\frac{E}{J}\right)_\psi, \tag{46}$$

where

$$\bar{\xi} = \operatorname{curl} \bar{v} = v_{2_x} - v_{1_y},$$
$$\Omega = H_{2_x} - H_{1_y}$$

and

$$h = \tfrac{1}{2}\rho v^2 + p + \tfrac{1}{2}\mu H^2.$$

5. REALIZATION OF THE CURVILINEAR SYSTEM

The curvilinear system (ϕ, ψ) is determined by the functions $f(\phi, \psi)$ and $g(\phi, \psi)$ according to the transformation in equation (29). These two functions define uniquely the tensor components E, F and G by virtue of equations (31) and (32). Conversely, for given tensor components E, F, G with $F \equiv 0$ the existence of two functions f and g satisfying equations (31) and (32) is not prior secured. We want to establish the necessary and sufficient condition on E and G for which f and g exist. To satisfy equations (31) and (32) identically we set

$$f_\phi = \sqrt{E}\cos\alpha, \quad f = -\sqrt{G}\sin\alpha,$$
$$g_\phi = \sqrt{E}\sin\alpha, \quad g = \sqrt{G}\cos\alpha, \tag{47}$$

where $\alpha(\phi, \psi)$ is still an arbitrary function of ϕ and ψ. We now apply Cartan's ω-symbolic [1] and introduce the Pfaffian forms

$$\omega_f = \sqrt{E}\cos\alpha\, d\phi - \sqrt{G}\sin\alpha\, d\psi,$$
$$\omega_g = \sqrt{E}\sin\alpha\, d\phi + \sqrt{G}\cos\alpha\, d\psi. \tag{48}$$

From the conditions that the exterior differentials of ω_f and ω_g vanish, i.e. that $d\omega_f = 0$ and $d\omega_g = 0$. We infer that $\alpha(\phi, \psi)$ is subject to the constraint

$$\alpha_\phi = -\frac{(\sqrt{E})_\psi}{\sqrt{G}}, \quad \alpha = \frac{(\sqrt{G})_\phi}{\sqrt{E}}. \tag{49}$$

Conversely, by virtue of constraint (49), the Pfaffian forms ω_f and ω_g are exact differentials. Similarly, equations (5.3) give rise to the Pfaffian form

$$\omega_\alpha = -\frac{(\sqrt{E})_\psi}{\sqrt{G}}\, d\phi + \frac{(\sqrt{G})_\phi}{\sqrt{E}}\, d\psi. \tag{50}$$

ω_α is an exact differential if (and only if) we have the condition

$$\left(\frac{(\sqrt{G})_\phi}{\sqrt{E}}\right)_\phi + \left(\frac{(\sqrt{E})_\psi}{\sqrt{G}}\right)_\psi = 0 \tag{51}$$

holds. Equation (51) is the constraint on E and G in question: it simply states that the Gaussian curvature of a planar surface vanishes.

6. RADIAL FLOWS

When a fluid flows in a radial direction, so that the pressure and velocity at any point in the flow varies with respect to the radial distance of that point from the central axis, then the flow is designated as a radial flow. The square of the element of arc length in polar coordinate system is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \tag{52}$$

and we have

$$\phi = \phi(r), \psi = \psi(\theta). \tag{53}$$

Using equations (53) in (30), we obtain

$$\mathrm{d}s^2 = E(\phi')^2\,\mathrm{d}r^2 + G(\psi')^2\,\mathrm{d}\theta^2. \tag{54}$$

Comparing equations (52) and (54), we find

$$E = \frac{1}{(\phi')^2}, \quad G = \frac{r_2^2}{(\psi')}. \tag{55}$$

From equations (55) and (35), we have

$$\psi' = \frac{kr}{\phi'} = A, \tag{56}$$

where A is an arbitrary constant. Using equations (55) and (56) in equations (45) and (46), we have

$$\mu\sigma = \frac{2}{A}, \quad \Omega = \frac{2k}{A}, \quad \xi = 0. \tag{57}$$

From the above equation, we find that the current density is constant in radial flows and the flow is irrotational. Substituting equations (56) and (55) in equation (43), we obtain

$$q = \exp\left(-\frac{\sigma}{\epsilon}\frac{r^2}{2A} + L\right). \tag{58}$$

Equation (58) shows that at any point within a conducting fluid, the charge decays very rapidly in an exponential manner. Electrostatic theory asserts that when static conditions prevail, the electric charge resides exclusively on the outer boundaries of a conductor. Thus, the charge within the conducting fluid decays exponentially at interior points, distributing itself on the outer boundaries.

It is not until $r = \infty$ that $\mathrm{e}^{-r/A}$, but in many cases this value is very nearly reached in so short a distance that there is a little error in considering the building-up of the charge to be instantaneous. The curves of decaying charge are plotted in Fig. 2. In this case $\lambda = 2A/\sigma r$, to fall to $1/\mathrm{e} = 0.368$ of its original value or to change by 0.632 of its total charge.

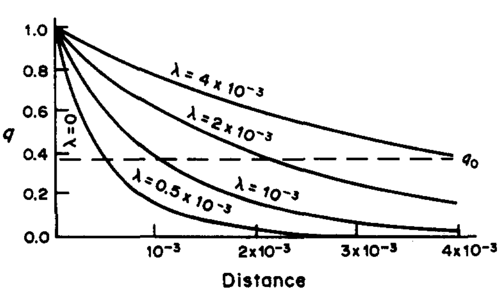

Fig. 2

Equations (40)–(42) give

$$vr = \frac{2}{\mu\sigma}, \quad H = \frac{\mu\sigma k}{2} r, \tag{59}$$

$$h = -\frac{\mu k^2 r^2}{A^2} - \left[\frac{A}{2\sigma} e^{2L} \sum_{n=0} \frac{\left(-\frac{\sigma}{A}\right)^n}{nn} r^{2n} \right] + M, \tag{60}$$

and

$$p = -\frac{\mu k^2 r^2}{A^2} - \frac{\rho}{2}\frac{A^2}{r^2} - \left[\frac{A}{2\sigma} e^{2L} \sum_{n=0} \frac{\left(-\frac{\sigma}{A}\right)^n}{nn} r^{2n} \right] + M. \tag{61}$$

Hence, from the equations (59)–(61), we find that the product of velocity and radial distance is always constant and further the pressure at any point varies with the square of radial distance from the central axis throughout the flow.

7. ANALYSIS FOR CIRCULATION-PRESERVING FLOWS

Equation (28) is itself necessary and sufficient condition for a steady motion to be circulation-preserving. We now establish the following theorem:

Theorem 1

The circulation preserving EMFD flows of steady vorticity are (i) motions with constant vorticity on which may be superposed a coplanar isochoric irrotational motion, which may be unsteady and (ii) motions whose streamlines are concentric circles, for which the velocity magnitude is

$$v = Ar \log r + Br + (C/r), \tag{62}$$

where A, B and C are constant.

To prove this proposition we let $\bar{s}$ and $\bar{n}$ be the unit tangent and normal vectors to the streamlines. The constant unit vector perpendicular to the plane of the motion is denoted by $\bar{b}$. The velocity is then given by

$$\bar{v} = v\bar{s}.$$

Taking the curl of the velocity vector, we have

$$\bar{\xi} = \text{curl}\, \bar{v} = \xi \bar{b}. \tag{63}$$

From the equation of continuity we obtain

$$\frac{\delta v}{\delta s} + k_n v = 0, \tag{64}$$

where the scalars k_s and k_n are identified as the curvatures of the vector lines of $\bar{s}$ and $\bar{n}$, respectively. Now from equation (64) we have

$$\text{curl}(\bar{\xi} x \bar{v}) = \text{grad}(v\xi) x \bar{n} + \bar{v} \times \text{curl}\, \bar{n} = v \frac{\delta}{\delta s} \bar{b} = 0 \tag{65}$$

By equation (1), we obtain

$$\text{curl}\, \xi = \frac{\delta \xi}{\delta n} \bar{s} - \frac{\delta \xi}{\delta s} \bar{n}. \tag{66}$$

Again since $\text{div}\,\text{curl}\, \xi = 0$, we have

$$\frac{\delta^2 \xi}{\delta s \delta n} - k_s \frac{\delta \xi}{\delta n} = 0. \tag{67}$$

Since $\delta\xi/\delta n$ is non-vanishing, we may write equations (66) and (67) in the respective forms

$$k_s = \frac{\delta\beta}{\delta n} \quad \text{and} \quad k_n = -\frac{\delta\beta}{\delta s}, \tag{68}$$

where

$$\beta = \log\left|\frac{\delta\xi}{\delta\beta}\right|.$$

Applying the commutation formulas in equations (68), we obtain

$$\frac{\delta k_s}{\delta s} + \frac{\delta k_n}{\delta n} = 0. \tag{69}$$

By equations (63)–(65), we have

$$k_n = 0, \tag{70}$$

so that, for rotational motion, $k_n = 0$ and the vector lines of $\bar{n}$ are straight lines. By equation (69), $\delta k_s/\delta s = 0$, so the vector lines of $\bar{s}$ are concentric circles. Here we write $k_s = 1/r$. By equations (65) and (66), v and ξ are functions of r only. Equation (65) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\xi}{\partial r}\right) = 0, \tag{71}$$

where, by equation (64)

$$\xi = \frac{1}{r}\frac{\partial}{\partial r}(rv). \tag{72}$$

By direct integration of equations (71) and (72), we have

$$v = Ar\log r \times Br + C/r.$$

This proves Theorem 1.

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